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Axial anomaly and meron configurations

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Abstract. A differential geometric interpretation of the axial anomaly is presented. This allows us to prove that for a two-dimensional Euclidean Schwinger model the axial anomaly is connected with the existence of meron configurations in the gauge sector of the theory. Arguments to support the same assertion in the case of a Euclidean four-dimensional theory are also given. Finally, several implications of the obtained results are presented.

1. Introduction

The purpose of this paper is to present and discuss a new perspective in the theory of the axial anomaly (for recent reviews of this topic and a large body of references see e.g. Crewther (1978a, b)). Firstly, we shall introduce a differential geometric, and at the same time extremal, interpretation of the axial anomaly condition, i.e. the divergence of the axial current is different from zero. (For a precise definition of these notions see § 2.) Then, in § 3, we shall apply these ideas to investigate the axial anomaly within the context of the two-dimensional Euclidean Schwinger model (Schwinger 1962). We shall prove, using two types of arguments, that in this case the axial anomaly is connected with the existence of meron configurations in the gauge sector of the Schwinger model.

In § 4 we shall try to extend the obtained results to the four-dimensional Euclidean QCD. We shall present arguments that support, in this case too, the previously established relationship between the axial anomaly and the meron configurations.

We acknowledge that t'Hooft (1976) has already suggested that the presence of self-dual (anti-self-dual) configurations in the gauge sector of a Euclidean QCD can be considered as a sufficient condition for the existence of the axial anomaly. Here we prove rigorously this assertion for a two-dimensional Euclidean Schwinger model and present plausibility arguments that suggest that the same holds in the four-dimensional case. The paper ends with a short discussion (§ 5).

2. Anomalies and immersions

Here we shall introduce some differential geometric results that will be instrumental in our subsequent derivations. First of all we shall define the notion of immersion of a Riemannian manifold into another one (Do Carmo 1976). Let M, N be two smooth (i.e. C^{∞}) Riemannian manifolds and $f:M\to N$ a smooth map from M into N. By definition f is an immersion if $\mathrm{d}f_x: T(M)_x \to T(N)_{f(x)}$ is injective for all $x \in M$. Here

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$T(M)_x$ ($T(N)_{f(x)}$) is the tangent space of M at the point x (tangent space of N at the point $f(x)$).

If ∇ ($\tilde{\nabla}$) is the covariant differentiation on M (N) then the well known Gauss theorem (Do Carmo 1976) implies

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) \tag{1}$$

for any couple X, Y of vector fields tangent to M. The quantity $\sigma(X, Y)$ is the second differential form of M in N. Now, if

$$H \simeq \mathrm{Tr}\, \sigma(X, Y) = 0 \tag{2}$$

the immersion f is called minimal. The trace H of the second differential form is called the mean curvature of the immersion f. (There exists another more rigorous definition of the minimality of an immersion in terms of the cohomology classes of the immersed manifold (Kuiper 1970). For our purposes it suffices to use the definition (2).)

The next step is to show the intimate connection between the divergence of a vector on a Riemannian manifold and the properties of the immersion(s) of that manifold. More precisely, if ξ is a vector field defined as

$$\xi : M \to T(N) \tag{3}$$

(with M, N Riemannian smooth manifolds) and ξ^T is the projection of ξ on $T(M)$, then one can prove the following relation (Hoffman and Spruck 1974):

$$\mathrm{div}_M\, \xi = \mathrm{div}_M\, \xi^T - (\xi, H). \tag{4}$$

Here (,) denotes the inner product in $T(M)$ induced from the inner product of M and H is the mean curvature of M (immersed in N). In the case when $\xi^T = 0$, then (4) becomes

$$\mathrm{div}_M\, \xi = -(\xi, H) \tag{4'}$$

and this relation will be our starting point for a differential geometric interpretation of the axial anomaly. Via relation (4′) we shall interpret the axial anomaly as expressing a non-minimal immersion of a certain manifold (into another one).

To conclude this section we shall present one more observation. As we have just affirmed, we shall interpret the axial anomaly (Adler 1969, Bell *et al* 1969, Bardeen 1974)

$$\partial_\mu J^5_\mu = cFF^* \tag{5}$$

as a particular realisation of the relation (4′). Here $F_{\mu\nu}$ is the strength tensor of the gauge field, $F^*_{\mu\nu}$ its dual ($F^*_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$), while J^5_μ is the non-symmetric gauge invariant axial current (which—as was strongly stressed by Crewther (1978a, b)—has nothing to do with the γ_5-transformation). The constant c in the relation (5) depends on the field model considered. (In this constant we have included the gauge trace $T(R)$ where $T(R)\delta^{ab} = \mathrm{Tr}\,\tau^a\tau^b$ and the matrices τ define a gauge group representation R to which the massless fermions belong.)

A necessary condition in order to perform the identification of (4′) with (5) is to consider the manifold M (in (4′)) as a spin manifold. Now, an oriented manifold M is a spin manifold if and only if $w_2(M) = 0$ (Milnor 1957). By $w_i(M) \in H^i(M, Z_2)$, $i \geq 1$, we understand the Stiefel–Whitney classes of M (Milnor 1957). (Notice that $H^i(M, Z_2)$ denotes the ith cohomology class—modulo Z_2—of the manifold M.) In the two-dimensional case (i.e. surfaces in R^3) it suffices if the manifolds are oriented. For instance, both S^2 (the two-dimensional unit sphere) and $T^2 = S^1 \times S^1$ (the two-dimensional torus) are oriented and therefore spin manifolds. Taking into account that the number of spin structures on an oriented surface is 4^g (where g is the genus of the surface, i.e. the number of linearly independent differentials which can be defined on that surface) we find that S^2 (which has genus $g = 0$) has only one spin structure, while T^2 (with $g = 1$) has four linearly independent spin structures. Generally, in order to calculate the number of (linearly independent) spin structures on a (spin) manifold, one has to determine the dimension of its first Stiefel–Whitney class $w_1(M)$ (Milnor 1957).

We are now in possession of (nearly) all facts necessary to investigate the problem we are interested in.

3. The two-dimensional Euclidean Schwinger model

We shall consider now the axial anomaly within the context of the two-dimensional Euclidean Schwinger model (Schwinger 1962). (For analyses of the structure of this model see e.g. Lowenstein and Swieca (1971) and Kogut and Susskind (1975).) In this case the analogue of the relation (5) is

$$\partial_i J_i^5 = c\epsilon_{ij} F^{ij}(x), \qquad x \in R^2,\ i = 1, 2, \tag{6}$$

and, a priori, there is no reason to start our discussion within the context of the self-dual (anti-self-dual) configurations. However, as we have already mentioned in the previous sections, we shall rigorously prove that the RHS of the relation (6) is different from zero only when one considers meron configurations in the gauge sector of this field model. After these introductory remarks we shall proceed now to present our proofs.

(a) Taking into consideration that the gauge group corresponding to the two-dimensional Euclidean Schwinger model is U(1) ($\simeq S^1$) one can say that the corresponding gauge potential $A_i(x)$, $x \in R^2$, describes the map

$$A_i(x): R^2 \to S^1, \qquad i = 1, 2, \tag{7}$$

or more precisely

$$A_i(x): M^2 \to S^1 \tag{8}$$

where M^2 is some two-dimensional spin manifold. Now, under the assumption that the maps (8) are described by regular homotopy classes, the obvious choice for M^2 is $T^2 = S^1 \times S^1$. With such a choice, (8) becomes

$$A_i(x): S^1 \times S^1 \to S^1 \tag{9}$$

i.e. essentially

$$A_i(x): S^1 \to S^1 \tag{10}$$

and it is characterised by $\pi_1(S^1) \simeq Z$. (Here $\pi^i(S^2)$ is the ith homotopy group of the unit sphere S^2.)

In differential geometric terms the map (10) can be understood as the Gauss map attached to the immersion $T^2 \to S^3$. If S is a surface immersed in R^3 or S^3 (Do Carmo 1976) the associated Gauss map is $S \to S^2$. For instance, both S^2 and $T^2 = S^1 \times S^1$ are minimally immersed in S^3 and the associated Gauss maps are

$$S^2 \to S^2 \tag{11}$$

and

$$T^2 = S^1 \times S^1 \to S^2 \qquad (\text{i.e. } S^1 \to S^1 \subset S^2). \tag{12}$$

Recently we showed (Tataru-Mihai 1978) that with these differential geometric notions it is possible to obtain a natural and rigorous interpretation of the classical solutions of the two-dimensional O(3)–σ–nonlinear model or of a SU(2) Euclidean Yang–Mills theory with axial symmetry ($A_\mu = A_\mu(r, t)$, $r = (\Sigma_{i=1}^3 x_i^2)^{1/2}$). For example, in the case of a two-dimensional O(3)–σ–nonlinear model the map (11) characterises the instanton solution, while the map (12) describes the two-dimensional merons. (Notice that the instantons correspond to the geodesic immersion (i.e. the associated second differential form is identically zero) $S^2 \to S^3$, while the merons are associated with the 'only' minimal immersion $T^2 \to S^3$. This difference in the type of the associated immersion explains the differences in the behaviour of the energies of these classical self-dual solutions.)

In the case of a SU(2) Yang–Mills theory with axial symmetry the space–time can be organised as the right-half plane of the complex plane $H_2 = \{z = r + it, r > 0\}$ or—via a Cayley transformation—as a unit disc $D^1 = \{w \mid w \in \mathbb{C}, |w| \leq 1\}$. The corresponding instantons are associated with the immersion $H_2 \to H_3$ (the three-dimensional real hyperbolic space), while the merons correspond to the automorphisms of the closed unit disc D^1.

Hence the differential geometric and homotopic characteristics of the two-dimensional Euclidean Schwinger model (i.e. the map (10)) are the same as those of a SU(2) Yang–Mills theory with axial symmetry (or those of a two-dimensional O(3)–σ–nonlinear model) when meron (i.e. toral) configurations are considered.

It is known that Witten (1977) has shown that a SU(2) Yang–Mills theory with axial symmetry can be viewed as an Abelian Higgs theory in D^1. Taking into account the relationship established above, this implies that the Schwinger model has to display a (dynamical) symmetry breaking. Such an effect has already been predicted through a different method by Lowenstein and Swieca (1971).

We have now arrived at the crucial point of our proof. Using the fact that the T^2 (i.e. the meron configuration) is not minimally immersed in R^3, we affirm that the axial anomaly (i.e. the RHS of the relation (6)) is determined by this non-minimal immersion. Glimm and Jaffe (1978) have investigated the meron solutions for a SU(2) Yang–Mills theory with axial symmetry and obtained for the strength tensor the expression

$$F^{ij}(x) \simeq \delta(r), \qquad r = \left(\sum_{i=1}^{2} x_i^2\right)^{1/2}. \tag{13}$$

Therefore

$$\partial_i J_i^5 \simeq \delta(r) \tag{14}$$

(notice that equations (13) and (14) are valid for the topological number $n = 1$).

On the other hand if one observes that T^2 can be written (in R^3) as

$$T^2 = \begin{matrix} (a + b \cos u) \cos v \\ (a + b \sin u) \sin v \\ b \sin u \end{matrix}$$

$$0 < b < a, \qquad (u, v) \in R^1 \times R^1 \tag{15}$$

then some simple algebra shows that

$$H_\xi^{T_2} \simeq \delta(r), \qquad r=(u^2+v^2)^{1/2} \tag{16}$$

i.e. a result consistent with equation (14).

Let us summarise now these partial results. Using differential geometric and homotopic arguments we showed that the geometry of the two-dimensional Euclidean Schwinger model is intimately connected with that of a SU(2) Yang–Mills theory with axial symmetry when a specific self-dual configuration (merons) is considered. The existence of a toral configuration in the gauge sector of the Schwinger model leads to a non-trivial RHS for the relation (6), i.e. can explain the possible axial anomaly.

(b) The second argument which will substantiate our assertion concerning a possible connection between the axial anomaly and the meron configurations (within the context of the considered field model) consists in an application of the spin index theorem (Shanahan 1978) to S^2 and T^2, respectively. (We acknowledge that the Atiyah–Singer index theorem has already been used in connection with the axial anomaly; see e.g. Crewther (1978a, b) for the relevant references. Here we shall present an application of the 'spin version' of the theorem.)

Essentially, to apply the spin index theorem to a given manifold M, one has to consider the action of an appropriately defined discrete group Γ on M and on its spin covering. (The spin covering is the double covering of the tangent bundle of M.) The spin index of M (consistent with the action of Γ) is obtained by determining the fixed point under Γ on M and on its spin covering.

We shall start with S^2. It is well known that the tangent bundle of S^2 is SO(3) and the double covering of SO(3) (i.e. the spin covering of S^2) is S^3 ($\mathrm{SO}(3)=S^3/(\pm 1)$). The discrete group to be considered is Z_2. If one takes $S^2=\mathbb{C}\cup\infty$ (i.e. the compactification of the complex plane $\mathbb{C}$) then the action of Z_2 on S^2 can be written as $gz=\lambda z$, $g\in Z_2$, $z\in S^2$. The constant λ is the second root of unity, i.e. $\lambda=e^{i\pi}$. The fixed points of the Z_2-action on S^2 are 0 and ∞ and, on S^3, they correspond to the point 1. The spin index theorem implies (see Shanahan (1978) for details)

$$\mathrm{Index}(g, S^2)=0, \qquad g\in Z_2. \tag{17}$$

We shall consider now the action of the group Z_2 on the two-torus $T^2=S^1\times S^1$. The group Z_2 acts on T^2 by $g(\theta,\phi)=(-\theta,-\phi)$, $g\in Z_2$, i.e. a 180° rotation of the torus about an axis of a coordinate system in R^3. The fixed points of this action are the corners of a square, e.g. $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π). This can be easily understood if one realises that T^2 can be represented as a lattice $T^2\simeq R^2/Z^2$. The spin covering of T^2 is $S^1\times S^1\times S^1$ and the spin index theorem leads to

$$\mathrm{Index}(g, T^2)=-2i. \tag{18}$$

Therefore, the axial anomaly leads to a non-trivial contribution only for T^2, i.e. for meron configurations. (The index theorem furnishes information on the RHS of the relation (6).)

As is well known, one can connect the condition $\operatorname{div}\boldsymbol{J}=0$ ($\boldsymbol{J}$ is a vector current) with the vanishing of the exterior derivative of an appropriately defined differential form (in terms of which one can write the flux of the vector current). Analogously, one can relate $\operatorname{div} J_i^5=\delta(r)$, $i=1,2$, to the exterior derivative of a differential form associated with the solid angle. (In R^n such an $(n-1)$-differential form associated with the solid angle is $\omega=\Sigma_{i=1}^n(J_i/r^{n-1})(\mathrm{d}x_i)^{-1}\wedge\mathrm{d}x$ where $J_i=x_i/r$ and $r=(\Sigma_{i=1}^n x_i^2)^{1/2}$, $\mathrm{d}x=\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_{i-1}\wedge\mathrm{d}x_{i+1}\wedge\ldots\wedge\mathrm{d}x_n$. The solid angle is determined by $\int_S\omega=\int_S(\boldsymbol{J}/r^{n-1})\,\mathrm{d}\boldsymbol{S}$ with S a surface in R^n. Taking into account that $x_i/r^2 = \partial(\ln r)/\partial x_i$, the exterior derivative of ω is $d\omega = \delta(r)$.) Although our intepretation is merely a 'mechanistic' one, it is interesting to point out that many years ago De Rham (1955) analysed the differential forms associated with the solid angle and observed that they are odd relative to a symmetry operation which interchanges the points on the spin covering of the manifold on which these forms exist.

4. The four-dimensional Euclidean QCD

In this section we shall try to extend the arguments presented above to the case of a four-dimensional Euclidean field theory including fermions and a non-Abelian Yang–Mills sector. We have proved (Tataru-Mihai 1979) that the four-dimensional merons (for a Euclidean Yang–Mills theory) can be understood via immersion,

$$S^1 \times S^3 \to S^5, \tag{19}$$

and the associated Gauss map is

$$S^1 \times S^3 \to S^4, \tag{20}$$

i.e. essentially

$$S^3 \to S^3 \subset S^4. \tag{21}$$

For the sake of completeness we write down also the instanton-type solution, i.e. the immersion

$$S^4 \to S^5 \tag{22}$$

and its associated Gauss map

$$S^4 \to S^4. \tag{23}$$

In contradistinction to S^4, the torus $S^1 \times S^3$ is not minimally immersed in R^5 and—via the results presented in § 3—it is reasonable to assume that this fact underlines the existence of the axial anomaly in a four-dimensional theory. Another (more speculative) plausibility argument to support this assumption is the following. In the two-dimensional case the axial anomaly is (with no doubt) associated with the existence of the merons (i.e. two-dimensional toral structures); on the other hand, as we have already pointed out, T^2 can accommodate four spin structures, i.e. it seems that the axial anomaly reveals the existence of a multiple spin structure on a given manifold. (There is some formal analogy with the anomaly observed by Gribov (1977) which reveals the existence of a multiple structure of the Yang–Mills vacuum.) In the four-dimensional case, both S^4 and $S^1 \times S^3$ are spin manifolds but $S^1 \times S^3$ admits more spin structures than the sphere S^4. Elsewhere I will try to present the calculation of the spin index for the torus $S^1 \times S^3$ (which can be considered as a Hopf manifold (Hirzebruch 1966)), i.e. a rigorous proof to support the assumption made in this section.

5. Discussion

To complete the analysis presented in the previous sections we shall discuss here some additional questions.

(i) The differential geometric interpretation of the axial anomaly proposed in this paper suggests also a natural way to avoid it. Conceiving an anomaly free model means to conceive a field theory in such a way that the meron configurations minimally immerse in the gauge group manifold. This observation could be used as one of the criteria to select field models of quarks and gluons.

(ii) Our results imply that at least in the two-dimensional case the gauge invariant operator (we use Crewther's notation (Crewther 1978a, b))

$$X(t)=\int J_0^5(x)\,\mathrm{d}^2x \tag{24}$$

is indeed connected with the topological charge, i.e.

$$X(t=+\infty)-X(t=-\infty)\simeq n \qquad (n \text{ is topological charge}). \tag{25}$$

(iii) A possible relation between the axial anomaly and the self-dual configurations has also been revealed by Kiskis (1977).

(iv) After a first version of this paper had been completed we became aware of a paper due to Kastrup (1978) where an extremal discussion of the axial anomaly (although with another language) is also attempted. However, there is no mention of a possible connection of the axial anomaly with the self-dual configurations in the gauge sector of a specific model.

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